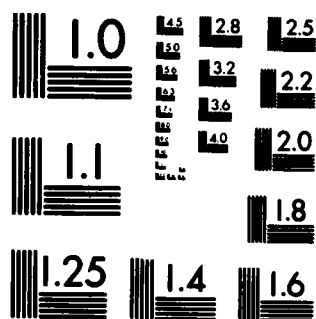


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MRC Technical Summary Report #2438

CONTINUUM MODELING OF TWO-PHASE FLOWS

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October 1982

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(Received September 29, 1982)

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CONTINUUM MODELING OF TWO-PHASE FLOWS

Donald A. Drew\*

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ABSTRACT

The equation of conservation of mass and momentum for two-phase flows are derived by applying a generic averaging process. The properties of this averaging process are discussed. Constitutive equations are proposed for the interfacial force. In addition, simple assumptions are made for the Reynolds stresses and the fluid viscosity. These assumptions are examined for transition layers.

AMS (MOS) Subject Classification: 76T05

Key Words: two-phase flows, transition layers, shocks, modeling

Work Unit Number 2 - Physical Mathematics

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Sponsored by the United States Army under Contract No. DAAG29-80-C-0041.

## CONTINUUM MODELING OF TWO-PHASE FLOWS

Donald A. Drew\*

### Introduction

The flow of two materials, one dispersed throughout the other, has received much attention in recent times. Unfortunately, at this time, there seems to be no set of equations which is regarded as fundamental, from which other models can be derived as approximations. (Consider the analogy with fluid mechanics, where the incompressible, inviscid equations are thought to be valid approximations outside of shear layers and boundary layers, when thermal and sonic effects are unimportant.)

Many researchers derive equations of motion by applying an averaging process to the microscopic equations of motion. The choice of averaging process is dictated by the taste of the researcher as well as the particular problem studied. In this paper, we give a derivation of the averaged equations by applying a generic average. The relation of the generic average to time- and space-averaging is discussed.

Once believable equations of motions have been formulated, it is natural to study their predictions in relatively simple flow situations. Often the constitutive assumptions used in the model are derived and/or tested on uniform flow situations. We give a discussion of transition layers in two-phase flows. A transition layer is a thin region where the concentration of one material changes rapidly in space. An example is the "interface" between

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a carbonated liquid and its foamy "head". An adequate description of these transitions provides a harsh test of the constitutive assumptions used in the model.

#### EQUATIONS OF MOTION

Each material is assumed to be a continuum, governed by the partial differential equations of continuum mechanics. The materials are separated by an interface, which is a surface. At the interface, jump conditions express the conditions of conservation of mass and momentum.

The equations of motion for each phase are (Truesdell and Toupin 1960)

- (1) conservation of mass

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \mathbf{v} = 0 \quad (1)$$

- (2) conservation of linear momentum

$$\frac{\partial \rho \mathbf{v}}{\partial t} + \nabla \cdot \rho \mathbf{v} \mathbf{v} = \nabla \cdot \mathbf{T} + \rho \mathbf{f} \quad (2)$$

valid in the interior of each phase. Here  $\rho$  denotes the density,  $\mathbf{v}$  the velocity,  $\mathbf{T}$  the stress tensor, and  $\mathbf{f}$  the body force density. Conservation of angular momentum becomes  $\mathbf{T} = \mathbf{T}^t$ , where  $t$  denotes the transposed. At the interface, the jump conditions are

- (1) jump condition for mass

$$[\rho(\mathbf{v} - \mathbf{v}_1) \cdot \mathbf{n}] = 0 \quad (3)$$

- (2) jump condition for momentum

$$[\rho \mathbf{v}(\mathbf{v} - \mathbf{v}_1) \cdot \mathbf{n} - \mathbf{T} \cdot \mathbf{n}] = 0 \quad (4)$$

Here  $[ ]$  denotes the jump across the interface,  $\mathbf{v}_1$  is the velocity of the interface, and  $\mathbf{n}$  is the unit normal (Aris 1962). We shall assume that  $\mathbf{n}$  points out of phase  $k$ , and that the jump between  $\mathbf{f}$  in phase  $k$  and  $\mathbf{f}$  in phase  $l$  defined by  $[\mathbf{f}] = \mathbf{f}^l - \mathbf{f}^k$ , where a superscript  $k$  denotes the limiting value from the phase  $k$  side. As a sign convention

for the curvature, we assume that  $\kappa$  is positive (concave) toward  $-n$ . The mass of the interface and surface stresses have been neglected. We do not discuss any thermodynamic relations in this paper.

Constitutive equations must be supplied to describe the behavior of each material involved. For example, if one material is an incompressible liquid, then specifying the value of  $\rho$ , and assuming  $T = -pI + \mu(Vv + (Vv)^t)$  determines the nature of the behavior of the fluid in that phase. Similar considerations are possible for solid particles or a gas. The resulting differential equations, along with the jump conditions, provide a fundamental description of the detailed or exact flow.

Usually, however, the details of the flow are not required. For most purposes of equipment or process design, averaged, or macroscopic flow information is sufficient. Fluctuations, or details in the flow must be resolved only to the extent that they affect the mean flow (like the Reynolds stresses affect the mean flow in a turbulent flow).

#### Averaging

In order to obtain equations which do not contain the details of the flow, it has become customary to apply some sort of averaging process. We present a generic averaging method, and its results.

Let  $\langle \rangle$  denote an averaging process so that if  $f(x,t)$  is an exact microscopic field, then  $\langle f \rangle(x,t)$  is the corresponding averaged field.

An averaging process assigns average values to certain variables. The ensemble, or set of possible outcomes, can be taken to be the possible flows in some apparatus where the initial and boundary conditions which are prescribed are equivalent in some sense. For example, for spherical particles, it may be necessary to give the statistical distribution of the positions and velocities of the centers of the particles at time  $t = 0$  such

that the average number density and average particle velocity is the same for all equivalent flows. We shall assume that there is some ensemble  $\Omega$ , with some appropriate weighting  $\mu(\omega)d\omega$  so that the average of  $f$  is given by

$$\langle f \rangle(x, t) = \int_{\Omega} f(x, t, \omega) \mu(\omega) d\omega.$$

Two cases can be discussed. If the flow is nearly steady, so that a time translation  $\tau$  makes no essential difference in the ensemble, it may be enough to consider the subset of the entire ensemble which consists of translations in time of amount  $\tau$ . We assign a weight  $\mu(\tau)$  to the likelihood of the flow whose outcome at time  $t$  is  $f(x, t-\tau)$ , where  $f(x, t)$  is the outcome at time  $t$  in some flow. The average of  $f$  is then taken to be

$$\langle f \rangle_t(x, t) = \int_{-\infty}^{\infty} f(x, t-\tau) \mu(\tau) d\tau.$$

This is classical time averaging; it is often used with

$$\mu(\tau) = \begin{cases} \frac{1}{T} & \text{if } 0 < \tau < T \\ 0 & \text{otherwise,} \end{cases}$$

although other averages are possible.

If there are no boundaries in the flow (that is, boundary effects are unimportant), then small spatial translations should make no difference in the ensemble. In analogy to the above, the average

$$\langle f \rangle_s(x, t) = \int_R f(x+s, t) \mu(s) ds$$

can be defined. This is the classical space average; it is often used with

$$\mu(s) = \begin{cases} \frac{1}{V} & \text{if } s \in V \\ 0 & \text{otherwise} \end{cases}$$



where  $V$  is some volume (for example, a sphere). Again, other averages are possible.

Thus, in some sense, ensemble averaging contains space and time averages as special cases. The averaging process is assumed to satisfy

$$\langle f+g \rangle = \langle f \rangle + \langle g \rangle \quad (5)$$

$$\langle \langle f \rangle g \rangle = \langle f \rangle \langle g \rangle \quad (6)$$

$$\langle c \rangle = c \quad (7)$$

$$\left\langle \frac{\partial f}{\partial t} \right\rangle = \frac{\partial}{\partial t} \langle f \rangle \quad (8)$$

$$\left\langle \frac{\partial f}{\partial x_1} \right\rangle = \frac{\partial}{\partial x_1} \langle f \rangle . \quad (9)$$

The first three of these relations are called Reynolds rules, the fourth is called Leibnitz' rule, and the fifth is called Gauss' rule.

In order to apply the average to the equations of motion for each phase, we introduce the phase function

$x_k(x, t)$  which is defined to be

$$x_k(x, t) = \begin{cases} 1 & \text{if } x \text{ is in phase } k \text{ at time } t \\ 0 & \text{otherwise .} \end{cases} \quad (10)$$

We shall deal with  $x_k$  as a generalized function, in particular in regard to differentiating it. Recall that a derivative of a generalized function can be defined in terms of a set of "test functions"  $\phi$ , which are "sufficiently smooth" and have compact support. Then  $\frac{\partial x_k}{\partial t}$  and  $\frac{\partial x_k}{\partial x_1}$  are defined by

$$\begin{aligned} \int_{R^3 \times R} \frac{\partial x_k}{\partial t} (x, t) \phi(x, t) dx dt &= \\ &= - \int_{R^3 \times R} x_k (x, t) \frac{\partial \phi}{\partial t} (x, t) dx dt , \end{aligned} \quad (11)$$

$$\int_{R^3 \times R} \frac{\partial x_k}{\partial x_i} (x, t) \phi(x, t) dx dt =$$

$$= - \int_{R^3 \times R} x_k(x, t) \frac{\partial \phi}{\partial x_i} (x, t) dx dt . \quad (12)$$

It can be shown that

$$\frac{\partial x_k}{\partial t} + v_i \cdot \nabla x_k = 0 \quad (13)$$

in the sense of generalized functions.

If  $f$  is smooth except at  $S$ , then  $f \nabla x_k$  is defined via

$$\int_{R^3 \times R} f \nabla x_k \phi dx dt = \int_{-\infty}^{\infty} \int_S n_k f^k \phi dS dt , \quad (14)$$

where  $n_k$  is the unit normal exterior to phase  $k$ , and  $f^k$  denotes the limiting value of  $f$  on the phase- $k$  side of  $S$ .

It is also clear that  $\nabla x_k$  is zero, except at the interface. Equation (14) describes the behavior of  $\nabla x_k$  at the interface. Note that it behaves as a "delta-function", picking out the interface  $S$ , and has the direction of the normal interior to phase  $k$ .

#### Averaged Equations

In order to derive averaged equations for the motion of each phase, we multiply the equation of conservation of mass valid in phase  $k$  (1) by  $x_k$  and average. Noting that

$$x_k \frac{\partial \rho}{\partial t} = \frac{\partial}{\partial t} x_k \rho - \rho \frac{\partial x_k}{\partial t} = \frac{\partial}{\partial t} x_k \rho + \rho v_i \cdot \nabla x_k \quad (15)$$

and

$$x_k \nabla \cdot \rho v = \nabla \cdot x_k \rho v - \rho v \cdot \nabla x_k , \quad (16)$$

we have

$$\frac{\partial}{\partial t} \langle x_k \rho \rangle + \nabla \cdot \langle x_k \rho v \rangle = \langle [\rho (v - v_i)]^k \cdot \nabla x_k \rangle . \quad (17)$$

Similar considerations for the momentum equations yield

$$\frac{\partial}{\partial t} \langle x_k \rho v \rangle + \nabla \cdot \langle x_k \rho v v \rangle = \nabla \cdot \langle x_k T \rangle + \langle x_k \rho f \rangle$$

$$+ \langle [\rho v (v - v_i) - T]^k \cdot \nabla x_k \rangle . \quad (18)$$

The terms

$$\langle [\rho(\mathbf{v} - \mathbf{v}_1)]^k \cdot \nabla \chi_k \rangle = \Gamma_k \quad (19)$$

and

$$\langle [\rho \mathbf{v}(\mathbf{v} - \mathbf{v}_1) - \mathbf{T}]^k \cdot \nabla \chi_k \rangle = M_k \quad (20)$$

are the interfacial source terms. As noted,  $\nabla \chi_k$  picks out the interface, and causes discontinuous quantities multiplying it to be evaluated on the phase- $k$  side of the interface.

The jump conditions come from equations (3) and (4). We have

$$\sum_{k=1}^2 \langle [\rho(\mathbf{v} - \mathbf{v}_1)]^k \cdot \nabla \chi_k \rangle = \sum_{k=1}^2 \Gamma_k = 0 \quad (21)$$

$$\sum_{k=1}^2 \langle [\rho \mathbf{v}(\mathbf{v} - \mathbf{v}_1) - \mathbf{T}]^k \cdot \nabla \chi_k \rangle = 0. \quad (22)$$

Applying a more specific averaging process (time averaging, for example) requires a different set of manipulations regarding the interfacial source terms (Anderson & Jackson 1967, Drew 1971, Ishii 1975, Delhay and Achard 1976). Almost all of the derivations for specific averaging processes seem to be more complicated than the above; however, the trade-off for the simple derivation is that all manipulations now involve generalized functions.

The volumetric concentration (or volume fraction, or relative residence time) of phase  $k$  is defined by

$$\alpha_k = \langle \chi_k \rangle. \quad (23)$$

We note that

$$\frac{\partial \alpha_k}{\partial t} = \langle \frac{\partial \chi_k}{\partial t} \rangle \quad (24)$$

and

$$\nabla \alpha_k = \langle \nabla \chi_k \rangle. \quad (25)$$

There are two types of averaged variables which are useful in two-phase mechanics, namely the phasic, or  $X_k$ -weighted average, and the mass-weighted average. Which is appropriate is suggested by the appearance of the quantity in the equation of motion. The phasic average of the variable  $\phi$  is defined by

$$\tilde{\phi}_k = \langle X_k \phi \rangle / \alpha_k \quad (26)$$

and the mass weighted average of the variable  $\psi$  is defined by

$$\hat{\psi}_k = \langle X_k \rho \psi \rangle / \alpha_k \hat{\rho}_k . \quad (27)$$

It is convenient to write the stresses  $\tilde{\tau}_k$  in terms of pressures plus extra stresses. Thus,

$$\tilde{\tau}_k = -\tilde{p}_k I + \tilde{\tau}_k . \quad (28)$$

It is expected that readers familiar with fluid dynamical concepts are familiar with the concept of pressure in fluids; in this case,  $\tilde{p}_k$  can be thought of as the average of the microscopic pressure. If one of the phases consists of solid particles, the concept is less familiar. In this case, the microscopic stress (involving small elastic deformations, for example) is thought of being made up of a spherical part (acting equally in all directions) plus an extra stress. The spherical part, when averaged, yields the pressure  $\tilde{p}_k$  in equation (28).

It has further become customary to separate various parts of the interfacial momentum transfer term. This is done by defining the interfacial velocity of the  $k^{\text{th}}$  phase by

$$\Gamma_k \mathbf{v}_{k,i} = \langle [\rho \mathbf{v}(\mathbf{v} - \mathbf{v}_1)]^k \cdot \nabla X_k \rangle , \quad (29)$$

and the interfacial pressure on the  $k^{\text{th}}$  phase by

$$p_{k,i} |\nabla q_k|^2 = \langle p^k \nabla X_k \cdot \nabla q_k \rangle . \quad (30)$$

Equation (30) is the dot product of  $\nabla q_k$  of the "standard" definition (Ishii 1975) of the interfacial pressure. The standard definition uses three

equations to define one scalar quantity, and cannot be a generally valid definition. Here the remaining part of the contribution of the pressure at the interface is lumped with the viscous stress contribution at the interface, and is treated through the use of a constitutive equation. Thus, we write

$$\mathbf{M}^k = \Gamma_k \nabla_{k,i} - p_{k,i} \nabla \mathbf{q}_k + \mathbf{M}_k^d, \quad (31)$$

where  $\mathbf{M}_k^d = \langle (p - p_{k,i})^k \nabla \mathbf{x}_k - \mathbf{x}_k^k \cdot \nabla \mathbf{x}_k \rangle$  is referred to as the interfacial force density, although it does not contain the effect of the average force on the interface due to the average interfacial pressure. The term  $-p_{k,i} \nabla \mathbf{q}_k$ , which does contain the force due to the average interfacial pressure, is sometimes referred to as the bouyant force. The reason for this terminology is, of course, that the buoyant force on an object is due to the distribution of the pressure of the surrounding fluid on its boundary.

With equations (23) and (26) - (31), the equations of motion (17) and (18) become

$$\begin{aligned} \frac{\partial \alpha_k \tilde{\rho}_k}{\partial t} + \nabla \cdot \alpha_k \tilde{\rho}_k \hat{\mathbf{v}}_k &= \Gamma_k \quad (32) \\ \frac{\partial \alpha_k \tilde{\rho}_k \hat{\mathbf{v}}_k}{\partial t} + \nabla \cdot \alpha_k \tilde{\rho}_k \hat{\mathbf{v}}_k \hat{\mathbf{v}}_k &= -\alpha_k \nabla \tilde{p}_k + \nabla \cdot \mathbf{q}_k (\tilde{\mathbf{x}}_k + \tilde{\mathbf{q}}_k) \\ &\quad + \Gamma_k \nabla_{k,i} + (p_{k,i} - \tilde{p}_k) \nabla \mathbf{q}_k \\ &\quad + \mathbf{M}_k^d. \end{aligned} \quad (33)$$

The jump conditions (21) and (22) are

$$\sum_{k=1}^2 \Gamma_k = 0 \quad (34)$$

$$\sum_{k=1}^2 [\Gamma_k \nabla_{k,i} + p_{k,i} \nabla \mathbf{q}_k + \mathbf{M}_k^d] = 0. \quad (35)$$

Adequate models for compressibility and phase change require consideration of thermodynamic processes. These are beyond the scope of this paper; therefore we shall restrict our attention to incompressible materials where no phase change occurs. Thus we assume that

$$\tilde{\rho}_k = \text{constant} \quad (36)$$

and

$$\Gamma_k = 0. \quad (37)$$

In order to simplify the notation, we shall drop all symbols denoting averaging.

#### CONSTITUTIVE EQUATIONS

In order to have a useable model, relations must be given which specify the stresses  $(\tau + \sigma_k)$ , the interfacial force density  $M_k^d$ , and the pressure differences  $p_k - p_{k,i}$ , consistent with the equations of motion and the jump conditions.

The fundamental process consists of proposing forms for the necessary terms within the framework of the principles of constitutive equations, finding solutions of the resulting equations, and verifying against experiments. The ideal end result of the process is a set of equations which could be used to predict the behavior of the two-phase flow, for example with a computer code. With the equations should come a set of conditions for the validity of the values of the constants and other functions used in the constitutive equations.

The stresses  $\tau_k + \sigma_k$ , the interfacial force density  $M_k^d$  and the pressure differences  $p_k - p_{k,i}$  are assumed to be functions of  $\alpha_k$ ,  $\partial \alpha_k / \partial t$ ,  $\nabla \alpha_k$ ,  $v_k$ ,  $\nabla v_k$ ,  $\partial v_k / \partial t$ , ... , where ... represents the material

properties, such as the viscosities and densities of the two materials, and other geometric parameters such as the average particle size, or the interfacial area density.

For concreteness, we shall refer to phase one as the particulate, or dispersed phase and include in that description solid particles, droplets, or bubbles. Phase two is then the continuous, or carrier phase, and can be liquid or gas. We shall denote

$$\alpha = \alpha_1 \quad (38)$$

so that

$$1 - \alpha = \alpha_2. \quad (39)$$

It is evident that both  $\alpha$  and  $1 - \alpha$  need not be included as independent variables in forming constitutive equations.

Drew and Lahey (1979) consider the general process of constructing constitutive equations. The simplest reasonable set of assumptions leads to models for the motions of the two materials which may be ill-posed. Drew (1982) gives a review of the state of affairs.

Essentially, the problem and the reason for its importance can be summarized in the following manner. The simplest model assumes that the interaction forces are due to viscous drag, and that pressure forces equilibrate across particles instantaneously. Thus,

$$\mathbf{M}_1^d = \mathbf{M}_2^d = \frac{3}{8} \alpha \rho_1 \frac{C_D}{r} |\mathbf{v}_1 - \mathbf{v}_2| (\mathbf{v}_2 - \mathbf{v}_1) \quad (40)$$

and

$$P_1 = P_2 = P_{1,1} = P_{2,1} = P. \quad (41)$$

If the stresses  $\tau_k + \sigma_k$  are ignored, the model has complex characteristics, and hence is ill-posed. See Ramshaw and Trapp (1978). The ill-posed nature of the model leads (theoretically) to solutions which grow rapidly on a small length scale. This leads some to conjecture that the model

is trying to resolve events on the microscale. The application of an average, however, is supposed to lose details of the flow; indeed, both (40) and (41) specifically ignore flow details on the scale of the particle size and smaller. The obvious conclusion is that equations (32 - 37), with assumptions (40), (41) and neglect of viscous forces lack some mechanism which is important on the particle scale. It is noteworthy that no difficulties arise in numerical simulations with the ill-posed system. The reason for this is that the instability inherent in the ill-posed system appears on a scale comparable to the particle radius to a power which depends on the exact form of the drag law used for  $C_D$ . Thus, if the mesh is not refined to the particle size, no instabilities will be seen. This suggests that computation with the ill-posed system with a reasonable mesh will most likely give good results.

The above argument indicates some difficulties for stability calculations. When is an instability not an instability? Presumably it is unobserved if its wavelength is too small. It would be satisfying to find the missing effect in the model and show how the model reduces to the isobaric, inviscid model mentioned above. There is a long list of candidates for the forgotten mechanism (Drew 1982) but no clear winner has emerged. The sensible approach seems to be to examine various models on a mesoscale, that is, on a scale which is small compared to the usual experimental verification flows (such as laminar settling in a still fluid), but large compared to the particle scale. These mesoscale calculations may indicate what sorts of terms are needed to give a valid description on a smaller scale.

There is another reason for seeking the more complete model. If the ill-posed model is the limit of some more complete model with some effects neglected, if the neglect can be done by a set of formal manipulations on the



more complete model, the result may indicate whether some well-posed model might do as well as the above ill-posed model. Moreover, it is always of interest to find reduced models which contain the same features as the original model.

We shall examine the effects of the viscous and Reynolds stresses on a mesoscale motion, namely transition layers in vertical flow. If the flow is vertical, and we denote the vertical velocities by

$$v_1 = u(z,t)k \quad (42)$$

$$v_2 = v(z,t)k, \quad (43)$$

we have

$$\frac{\partial \alpha}{\partial t} + \frac{\partial \alpha_1}{\partial z} = 0 \quad (44)$$

$$\alpha_1 + (1-\alpha)v = j(t) \quad (45)$$

$$\begin{aligned} \alpha \rho_1 \left( \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial z} \right) = & -\alpha \frac{\partial p}{\partial z} + \alpha b(v-u) - \\ & - \alpha \rho_1 g + \frac{\partial}{\partial z} \alpha \sigma_1 \end{aligned} \quad (46)$$

$$\begin{aligned} (1-\alpha) \rho_2 \left( \frac{\partial v}{\partial t} + v \frac{\partial v}{\partial z} \right) = & -(1-\alpha) \frac{\partial p}{\partial z} + \alpha b(u-v) - (1-\alpha) \rho_2 g + \\ & + \frac{\partial}{\partial z} (1-\alpha) \sigma_2 + \frac{\partial}{\partial z} (1-\alpha) \mu \frac{\partial v}{\partial z}. \end{aligned} \quad (47)$$

The function  $j(t)$  is the volumetric flux, and will be taken to be constant. The stress models are

$$\tau_1 = 0 \quad (48)$$

$$\tau_2 = \mu \frac{\partial v}{\partial z} \quad (49)$$

to represent the viscosity of the fluid. The particles are assumed to be inviscid. In addition, the Reynolds stress terms  $\sigma_k$  will be taken to be constants.

For problems of sedimentation and transitions in fluidized beds, we shall assume that  $j(t) = \text{constant}$  for  $t > 0$ .

#### Kinematic Waves

If we ignore inertia in equations (46) and (47) and set  $\sigma_1 = \sigma_2 = 0$  and  $\mu = 0$ , we have a model which reduces to a one-dimensional scalar conservation law. If the pressure is eliminated from the resulting momentum equations, we have

$$v-u = \frac{(1-\alpha)(\rho_1-\rho_2)g}{b} . \quad (50)$$

Using (45) gives

$$u = j - (1-\alpha)(v-u) .$$

Hence

$$\alpha u = \alpha j - \frac{\alpha(1-\alpha)^2(\rho_1-\rho_2)g}{b} = f(\alpha) . \quad (51)$$

Equation (44) with  $\alpha u$  given by (51) is a scalar conservation law for  $\alpha(z,t)$ .

Solutions can be found by the method of characteristics; on the characteristics,

$$\frac{d\alpha}{dt} = 0 . \quad (52)$$

$$\frac{dz}{dt} = f'(\alpha) = \text{constant} . \quad (53)$$

As long as no discontinuities occur,  $\alpha$  is constant on characteristics.

If discontinuities occur, they must propagate at a speed given by

$$s = \frac{[f]}{[\alpha]} , \quad (54)$$

where  $[\phi]$  denotes the jump in the quantity  $\phi$  across the discontinuity. A discontinuity, or shock will persist if characteristics tend to come into the shock. If characteristics diverge from the shock, then it will smooth out. In Figure 1, a shock from  $(\alpha_-, f(\alpha_-))$  to  $(\alpha_+, f(\alpha_+))$  is not stable if

the region where  $\alpha_+$  occurs is above the shock. This is because the characteristics at  $\alpha_+$  are moving faster than the shock. Note that contact discontinuities are possible with this model.

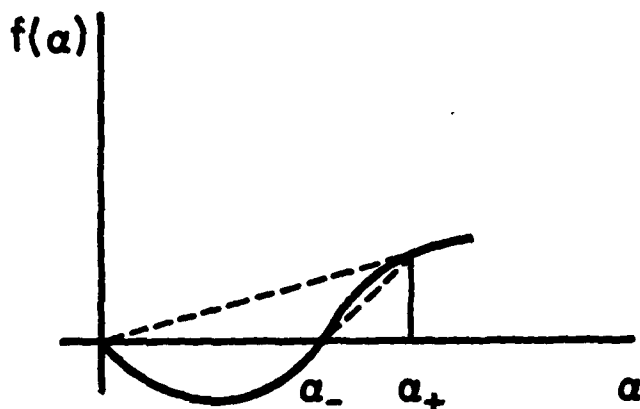


Figure 1

For fluidization,  $j > 0$ . In order to describe a transition in fluidization, we assume that the bed is fluidized at  $t = 0^-$  with a concentration  $\alpha_+$  which corresponds to some value of the volumetric flow rate  $j_+$ . At time  $t = 0$ , the volumetric flow rate is instantaneously changed to  $j$ . The flow-concentration diagram is given in Figure 1. The simplest situation is depicted, consisting of an upward traveling bottom shock (transition) and an upward traveling top shock.

Figure 2 shows the concentration at some time  $t_1$ , and Figure 3 shows the solution in various regions of the  $t - z$  plane.

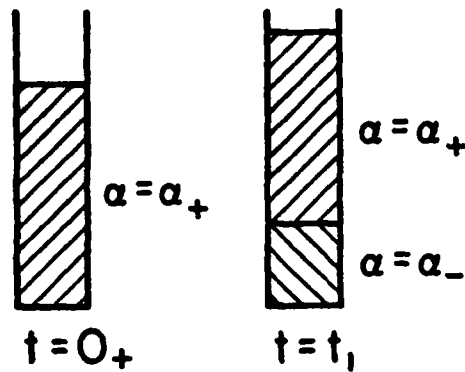


Figure 2

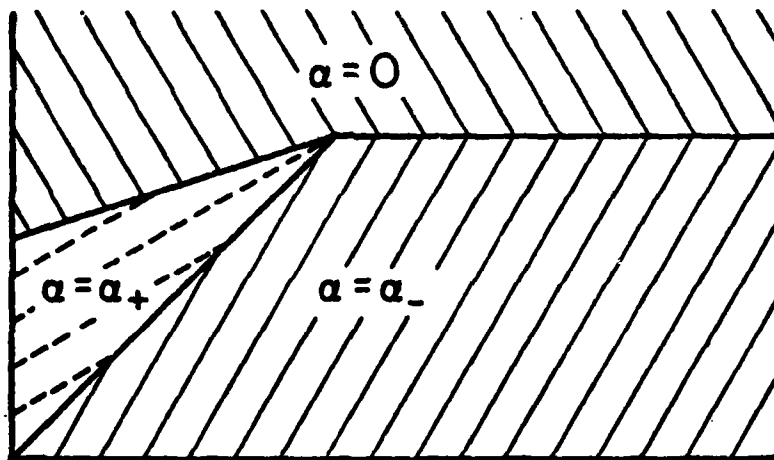


Figure 3

#### Diffusional Regularization

One way to understand shocks is to include in the model some means of smearing them out. This can be done in the present problem by including the diffusion terms. We continue to ignore inertia and viscosity. Repeating the procedure outlined above yields

$$\alpha_t = f(\alpha) - D \frac{\partial \alpha}{\partial z}, \quad (55)$$

where

$$D = \frac{(1-\alpha)[(1-\alpha)\sigma_1 + \alpha\sigma_2]}{b} . \quad (56)$$

The equation for  $\alpha$  is a nonlinear diffusion equation

$$\frac{\partial \alpha}{\partial t} + \frac{\partial f(\alpha)}{\partial z} = \frac{\partial}{\partial t} \left( D \frac{\partial \alpha}{\partial z} \right) . \quad (57)$$

In order to examine the transition, we let

$$\alpha = \alpha(z - st) \quad (58)$$

where  $s$  is the speed of propagation of the traveling wave representing the transition. We obtain

$$\frac{d}{d\xi} [f(\alpha) - s\alpha] = \frac{d}{d\xi} \left( D \frac{d\alpha}{d\xi} \right) . \quad (59)$$

Integrating from  $-\infty$  to  $\xi$  gives

$$f(\alpha) - [f(\alpha_-) + s(\alpha - \alpha_-)] = D \frac{d\alpha}{d\xi} . \quad (60)$$

The quantity in the bracket is the equation for the chord (in Figure 1 for example), and  $f(\alpha)$  is the flow-concentration curve there. If the curve lies above the chord, then  $d\alpha/d\xi$  is positive. If the curve lies below the chord, then  $d\alpha/d\xi$  is negative. The transitions go from  $\alpha_-$  to  $\alpha_+$  where  $\alpha_+$  is the value of  $\alpha$  where the left hand side of (60) is zero, that is, where the curve and chord intersect. If  $D$  is small, the transition region is thin. The diffusional regularization corresponds to the results obtained from shock stability considerations.

Inclusion of the inertia of both phases complicates the situation immensely. It can be shown that the thing which corresponds to the shock in the kinematic wave model is not a shock in the model which includes inertia.

In order to study the transition, assume that  $\alpha$ ,  $u$ ,  $v$  and  $p$  are functions of  $\xi = z - st$ , where  $s$  is the speed of the transition wave, and  $\alpha \rightarrow \alpha_-$ ,  $u \rightarrow u_-$ ,  $v \rightarrow v_-$  as  $\xi \rightarrow -\infty$ ,  $\alpha \rightarrow \alpha_+$ ,  $u \rightarrow u_+$ ,  $v \rightarrow v_+$  as  $\xi \rightarrow \infty$ .  $\alpha_{\pm}$ ,  $u_{\pm}$  and  $v_{\pm}$  are related to  $s$  by (45) and (54). Substituting in the equations (44 - 47) and eliminating  $p$ ,  $u$ , and  $v$  gives

$$\mu(1-\alpha_-)(v_- - s) \frac{\alpha}{(1-\alpha)^2} \alpha'' + \quad (61)$$

$$+ \hat{\beta}(\alpha) \alpha' + \hat{g}(\alpha) = 0$$

where

$$\begin{aligned} \hat{\beta}(\alpha) = \sigma_1 + \sigma_2 \frac{\alpha}{1-\alpha} - \frac{\alpha_-^2 \rho_1 (u_- - s)^2}{\alpha^2} - \\ - \frac{\alpha}{(1-\alpha)^3} \rho_2 (1-\alpha_-)^2 (v_- - s)^2 \end{aligned} \quad (62a)$$

and

$$\hat{g}(\alpha) = - \frac{b}{(1-\alpha)^2} [f(\alpha) - \alpha_- u_- s (\alpha - \alpha_-)] . \quad (62b)$$

Note that if  $\mu = 0$ , and we wish to retain the picture given by the characteristics of the scalar conservation law, we can do so only if

$$\hat{\beta}(\alpha) > 0 \quad (63)$$

for  $\alpha$  between  $\alpha_-$  and  $\alpha_+$ . In a sense, equation (63) suggests that inertia will be unimportant for transitions if diffusion is sufficiently large; sufficiently large means so that equation (63) is satisfied.

Note from equation (61) with  $\mu = 0$  and  $\sigma_1 = \sigma_2 = 0$ , that inertia without diffusion gives results opposite to the results from the scalar conservation law.

#### The Effect of Viscosity

In equation (61), let  $\alpha' = w(\alpha)$ . Then  $\alpha'' = w \frac{dw}{d\alpha}$ , and equation (61) becomes

$$\begin{aligned} \mu(1-\alpha_-)(v_- - s) \frac{\alpha}{(1-\alpha)^2} \frac{d(w^2/2)}{d\alpha} + \mu(1-\alpha_-)(v_- - s) \frac{\alpha}{(1-\alpha)^3} w^2 \\ + \hat{\beta}(\alpha)w + \hat{g}(\alpha) = 0 . \end{aligned} \quad (64)$$

Let us now define  $G(\alpha)$  by

$$G'(\alpha) = g(\alpha) \quad (65)$$

and

$$\begin{aligned} G(\alpha_-) &= 0 \quad \text{if } g(\alpha) > 0 \quad \text{for } \alpha_- < \alpha < \alpha_+ \\ G(\alpha_+) &= 0 \quad \text{if } g(\alpha) < 0 \quad \text{for } \alpha_- < \alpha < \alpha_+ . \end{aligned} \quad (66)$$

(We also assume that  $(1-\alpha)(v-s) > 0$ .) Equation (64) becomes

$$\frac{d}{d\alpha} \left[ \frac{1}{2} \left( \frac{w}{1-\alpha} \right)^2 + G(\alpha) \right] = -\beta(\alpha)w . \quad (67)$$

The curves  $H(\alpha, w) = \frac{1}{2} \left( \frac{w}{1-\alpha} \right)^2 + G(\alpha)$  are closed curves centered at  $\alpha_-$  if  $g(\alpha) > 0$ , and  $\alpha_+$  if  $g(\alpha) < 0$ . See Figure 4. On a trajectory leaving  $\alpha_+$  (if  $g(\alpha) > 0$ ) or  $\alpha_-$  (if  $g(\alpha) < 0$ ), the function  $\alpha(\xi)$  satisfies

$$\frac{d}{d\xi} H(\alpha, w) = -\beta(\alpha)w^2 \quad (68)$$

which is negative if  $\beta(\alpha) > 0$ . Thus  $H$  decreases on a trajectory, giving the transition from  $\alpha_+$  to  $\alpha_-$  (if  $g(\alpha) > 0$ ) or from  $\alpha_-$  to  $\alpha_+$  (if  $g(\alpha) < 0$ ). Since

$$g(\alpha) = - \frac{b}{\alpha(1-\alpha)^2, \mu(1-\alpha_-)(v-s)} (f(\alpha) - \alpha_- u_- - s(\alpha - \alpha_-)) ,$$

we see that if the curve  $f$  is above the chord,  $s(\alpha - \alpha_-) + \alpha_- u_-$ , then  $g(\alpha) < 0$ , and this type of transition can occur from  $\alpha_-$  to  $\alpha_+$ . On the other hand, if the curve  $f(\alpha)$  is below the chord  $s(\alpha - \alpha_-) + \alpha_- u_-$ , then  $g(\alpha) > 0$ , and the transition can occur from  $\alpha_+$  to  $\alpha_-$ . This again agrees with the picture given by the characteristics for the one-dimensional conservation law.

### Conclusion

There are two distinct features of two-phase flow modeling addressed in this paper, namely the averaging process, and transition layers.

The generic averaging process includes certain classical averages as special cases and gives the same results as these classical averages when applied to the equations of motion.

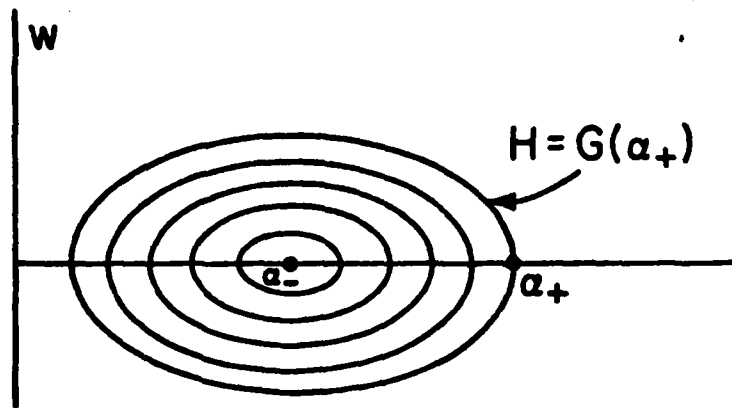


Figure 4

Transition layers are often observed in two-phase flows, and various models for them have been used. The simplest model, that of the kinematic wave, describes the situation when inertia, viscosity and diffusivity are negligible, but diffusivity dominates over inertia.



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REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM	
1. REPORT NUMBER #2438	2. GOVT ACCESSION NO. AD-A124 385	3. RECIPIENT'S CATALOG NUMBER	
4. TITLE (and Subtitle) Continuum Modeling of Two-Phase Flows		5. TYPE OF REPORT & PERIOD COVERED Summary Report - no specific reporting period	
		6. PERFORMING ORG. REPORT NUMBER	
7. AUTHOR(s) Donald A. Drew		8. CONTRACT OR GRANT NUMBER(s) DAAG29-80-C-0041	
9. PERFORMING ORGANIZATION NAME AND ADDRESS Mathematics Research Center, University of 610 Walnut Street Madison, Wisconsin 53706		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS Work Unit Number 2 - Physical Mathematics	
11. CONTROLLING OFFICE NAME AND ADDRESS U. S. Army Research Office P. O. Box 12211 Research Triangle Park, North Carolina 27709		12. REPORT DATE October 1982	
		13. NUMBER OF PAGES 21	
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)		15. SECURITY CLASS. (of this report) UNCLASSIFIED	
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE	
16. DISTRIBUTION STATEMENT (of this Report) Approved for public release; distribution unlimited.			
17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)			
18. SUPPLEMENTARY NOTES			
19. KEY WORDS (Continue on reverse side if necessary and identify by block number) two-phase flows, transition layers, shocks, modeling			
20. ABSTRACT (Continue on reverse side if necessary and identify by block number) The equation of conservation of mass and momentum for two-phase flows are derived by applying a generic averaging process. The properties of this averaging process are discussed. Constitutive equations are proposed for the interfacial force. In addition, simple assumptions are made for the Reynolds stresses and the fluid viscosity. These assumptions are examined for transition layers.			

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